

This chapter will be devoted to the kinematics of robot manipulators, which means studying geometrically the motion of the manipulator links and/or objects related to the manipulation task in terms of position, velocity, and acceleration.

First, to express the position and orientation of a rigid object, the method of assigning a coordinate frame to the object will be explained. Homogeneous transforms, which are convenient for describing the relations among many objects, will be introduced. Then an expression for finding the end effector's position in space from the description of the manipulator mechanism and its joint displacements will be derived. Methods for obtaining the joint displacements that realize a given end-effector position will also be discussed.

Next, the Jacobian matrix describing the relation between the joint's velocity and the end effector's velocity will be introduced. We will consider the problem of finding a joint velocity that achieves a given end effector velocity and discuss singular configurations and an application of the Jacobian matrix to some problems of statics.

2.1 Position and Orientation of Objects

2.1.1 Object Coordinate Frame

The first thing we have to do to analyze a manipulator is represent mathematically the position and orientation of the manipulator itself, the tool it holds, and the objects on which the robot works in three-dimensional space. For this purpose we adopt a method generally used in mechanics¹: we attach an orthogonal coordinate frame to each object and express its position and orientation by the position of the origin and the directions of the three axes of the attached frame relative to a given reference orthogonal coordinate frame. The former is called the *object frame* and the latter the *reference frame*.

Consider the object shown in figure 2.1. The reference frame is denoted by Σ_A , its origin by O_A , and the three coordinate axes by X_A , Y_A , and Z_A . Similarly, the object frame is denoted by Σ_B , its origin by O_B , and the three axes by X_B , Y_B , and Z_B . The vector from O_A to O_B (i.e., the position vector of O_B relative to O_A), expressed in Σ_A , is denoted as ${}^A\mathbf{p}_B$.^{*} The unit vectors

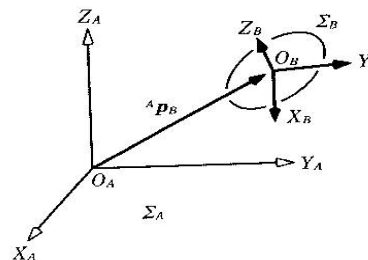


Figure 2.1
Reference frame and object frame.

in the directions of X_B , Y_B , and Z_B , expressed in Σ_A , are denoted ${}^A\mathbf{x}_B$, ${}^A\mathbf{y}_B$, and ${}^A\mathbf{z}_B$. Then the position of the object is represented by ${}^A\mathbf{p}_B$, and its orientation is represented by $\{{}^A\mathbf{x}_B, {}^A\mathbf{y}_B, {}^A\mathbf{z}_B\}$. The superscript A on the left means that the vector is expressed in the frame Σ_A . Hereafter, all frames denoted by Σ are right-hand orthogonal coordinate frames.

2.1.2 Rotation Matrix

It was established in subsection 2.1.1 that the orientation of an object can be specified by the three vectors $\{{}^A\mathbf{x}_B, {}^A\mathbf{y}_B, {}^A\mathbf{z}_B\}$. It is often convenient, however, to use a matrix defined by

$${}^A\mathbf{R}_B = [{}^A\mathbf{x}_B \ {}^A\mathbf{y}_B \ {}^A\mathbf{z}_B] \tag{2.1}$$

instead of the three vectors. This matrix ${}^A\mathbf{R}_B$, which can be regarded as describing the rotational part of the relative displacement of frame Σ_B from frame Σ_A , is called the *rotation matrix*.

Example 2.1 Consider the problem of expressing the position and orientation of a two-fingered hand. The hand frame Σ_H is attached to the hand as shown in figure 2.2a. In words, the origin O_H is taken to be at the midpoint of the two fingers. The Z_H axis is in the approach direction of the hand. The Y_H axis is in one of the two directions in the plane including the two fingers and normal to Z_H . The X_H axis is assigned so as to be normal to Z_H and Y_H , making Σ_H a right-hand coordinate system. We assume that, starting from the state (shown in figure 2.2b) where the hand frame Σ_H agrees with the reference frame Σ_A , the hand is rotated 90° around the axis that goes through $[0,0,2]^T$ of Σ_A and parallel to X_A . The hand's position

^{*} Vectors and matrices are denoted by boldface italic letters, the only exception being identity matrices and zero matrices; those will be denoted by \mathbf{I} and \mathbf{O} , respectively.

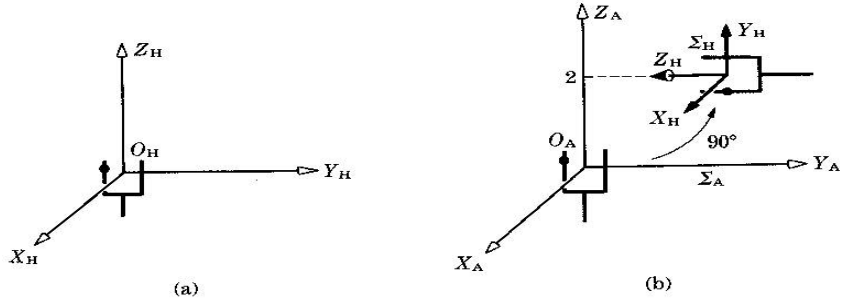


Figure 2.2 Representation of position and orientation of a hand. (The black dot on one finger distinguishes it from the other finger.) (a) Hand frame Σ_H . (b) Relation between Σ_A and Σ_H .

and orientation after rotation are given by

$${}^A p_H = [0, 2, 2]^T$$

and

$${}^A R_H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}. \square^*$$

Some properties of the rotation matrix will now be given. Since ${}^A x_B, {}^A y_B$, and ${}^A z_B$ are unit vectors orthogonal to one another, they satisfy

$$\begin{aligned} ({}^A x_B)^T {}^A x_B &= 1, \\ ({}^A y_B)^T {}^A y_B &= 1, \\ ({}^A z_B)^T {}^A z_B &= 1, \\ ({}^A x_B)^T {}^A y_B &= 0, \\ ({}^A y_B)^T {}^A z_B &= 0, \\ ({}^A z_B)^T {}^A x_B &= 0. \end{aligned} \tag{2.2}$$

Hence ${}^A R_B$ satisfies

$$({}^A R_B)^T ({}^A R_B) = I_3, \tag{2.3}$$

* An open square (\square) denotes the end of an example.

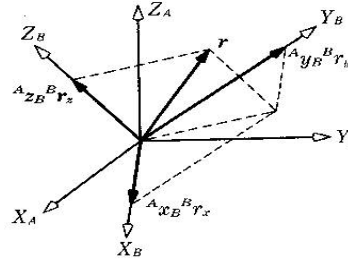


Figure 2.3 Interpretation of ${}^A r = {}^A R_B {}^B r$.

where I_3 is the 3×3 identity matrix. From equation 2.3, we have

$${}^A R_B^{-1} = ({}^A R_B)^T, \tag{2.4}$$

where R^{-1} denotes the inverse of a matrix R . Therefore, the rotation matrix ${}^A R_B$ is an orthogonal matrix.

Consider the case where the origins of Σ_A and Σ_B are at the same point, as shown in figure 2.3, and assume that a vector r is described by

$${}^A r = [{}^A r_x, {}^A r_y, {}^A r_z]^T$$

when expressed in Σ_A , and by

$${}^B r = [{}^B r_x, {}^B r_y, {}^B r_z]^T$$

when expressed in Σ_B . Then we have

$${}^A r = {}^A x_B {}^B r_x + {}^A y_B {}^B r_y + {}^A z_B {}^B r_z,$$

or

$${}^A r = {}^A R_B {}^B r. \tag{2.5}$$

Replacing Σ_A and Σ_B yields

$${}^B r = {}^B R_A {}^A r. \tag{2.6}$$

From equations 2.5 and 2.6 we have

$${}^A r = {}^A R_B {}^B R_A {}^A r. \tag{2.7}$$

Since equation 2.7 holds for any ${}^A r$, we obtain

$${}^A R_B {}^B R_A = \mathbf{I}_3, \quad (2.8)$$

or

$${}^B R_A = ({}^A R_B)^{-1} = ({}^A R_B)^T. \quad (2.9)$$

Further, consider the case where the origin of a third frame, Σ_C , coincides with the common origin of Σ_A and Σ_B , and assume that r is described by ${}^C r$ when expressed by Σ_C . Then

$${}^B r = {}^B R_C {}^C r, \quad (2.10)$$

$${}^A r = {}^A R_C {}^C r. \quad (2.11)$$

Hence, from equations 2.5, 2.10, and 2.11, we have

$${}^A R_C {}^C r = {}^A R_B {}^B R_C {}^C r. \quad (2.12)$$

Since equation 2.12 holds for any ${}^C r$, we obtain the relation

$${}^A R_C = {}^A R_B {}^B R_C. \quad (2.13)$$

An expression of ${}^A R_B$ different from equation 2.1 can be obtained using equation 2.13. First, from equation 2.13 we have

$${}^A R_B = ({}^C R_A)^T {}^C R_B. \quad (2.14)$$

Let ${}^c x_A$, ${}^c y_A$, and ${}^c z_A$ denote the unit vectors in the directions X_A , Y_A , and Z_A , and let ${}^c x_B$, ${}^c y_B$, and ${}^c z_B$ denote those in the directions X_B , Y_B , and Z_B , both expressed in Σ_C . Then ${}^A R_B$ can also be written as

$${}^A R_B = \begin{bmatrix} ({}^c x_A)^T {}^c x_B & ({}^c x_A)^T {}^c y_B & ({}^c x_A)^T {}^c z_B \\ ({}^c y_A)^T {}^c x_B & ({}^c y_A)^T {}^c y_B & ({}^c y_A)^T {}^c z_B \\ ({}^c z_A)^T {}^c x_B & ({}^c z_A)^T {}^c y_B & ({}^c z_A)^T {}^c z_B \end{bmatrix}. \quad (2.15)$$

Note that, for example, the (1,1) element, $({}^c x_A)^T {}^c x_B$, is the cosine of the angle between the two vectors ${}^c x_A$ and ${}^c x_B$. The same holds for all other elements. For this reason, the rotation matrix ${}^A R_B$ is also called the *directional cosine matrix*. The value of ${}^A R_B$ is, of course, independent of the choice of Σ_C .

Although ${}^A R_B$ has nine variables, they always satisfy equation 2.2. Hence, the representation by ${}^A R_B$ is redundant. When any two of the three vectors ${}^A x_B$, ${}^A y_B$, and ${}^A z_B$ are given, we can obtain the last vector using equation

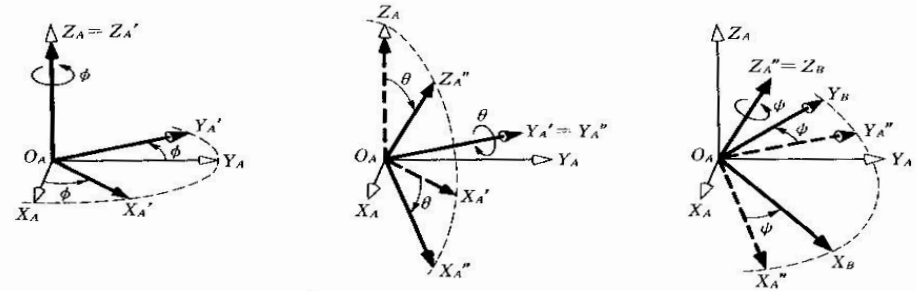


Figure 2.4
Euler angles (ϕ, θ, ψ) .

2.2.* For instance, the pair $\{{}^A z_B, {}^A x_B\}$, which has six variables, can represent the orientation. So long as the elements of the rotation matrix themselves are used, there seems to be no way of representing the orientation by five or fewer variables. On the other hand, intuitively speaking, since equation 2.2 provides six relations to nine variables, the orientation should be describable by three variables, just as the position is. In fact, Euler angles and roll-pitch-yaw angles are convenient established representations. Both of them regard the object frame Σ_B as a result of three sequential rotations about some fixed axes from the reference frame Σ_A , and represent the orientation of Σ_B by the set of three rotational angles.

2.1.3 Euler Angles

There are several definitions of Euler angles. A rather common definition, illustrated in figure 2.4, is adopted here:

- (i) The coordinate frame obtained from a rotation about the Z_A axis by an angle ϕ of a frame initially coincident with Σ_A is named $\Sigma_{A'}$ ($O_A - X_{A'} Y_{A'} Z_{A'}$).
- (ii) The frame obtained from $\Sigma_{A'}$ by a rotation about $Y_{A'}$ by an angle θ is named $\Sigma_{A''}$ ($O_A - X_{A''} Y_{A''} Z_{A''}$).
- (iii) The frame obtained from $\Sigma_{A''}$ by a rotation about $Z_{A''}$ by an angle ψ is named Σ_B .

Thus, the orientation of Σ_B with respect to Σ_A can be represented by the

* Strictly speaking, determining the last vector requires, besides equation 2.2, the condition that ${}^A x_B$, ${}^A y_B$, and ${}^A z_B$ form a right-hand system. See exercise 2.17.

set of three angles (ϕ, θ, ψ) , which are called the *Euler angles*. The relation between the Euler angles and the rotation matrix ${}^A\mathbf{R}_B$ is as follows.

First, from figure 2.4a, the relation between Σ_A and $\Sigma_{A'}$ is given by

$${}^A\mathbf{R}_{A'} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.16)$$

Similarly, the rotation matrix ${}^{A'}\mathbf{R}_{A''}$ between $\Sigma_{A'}$ and $\Sigma_{A''}$, and the rotation matrix ${}^{A''}\mathbf{R}_B$ between $\Sigma_{A''}$ and Σ_B , are given by

$${}^{A'}\mathbf{R}_{A''} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad (2.17)$$

and

$${}^{A''}\mathbf{R}_B = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.18)$$

From equation 2.13, the final rotation matrix ${}^A\mathbf{R}_B$ resulting from the three rotations is obtained from

$${}^A\mathbf{R}_B = {}^A\mathbf{R}_{A'} {}^{A'}\mathbf{R}_{A''} {}^{A''}\mathbf{R}_B, \quad (2.19)$$

which is

$${}^A\mathbf{R}_B = \begin{bmatrix} \cos\phi \cos\theta \cos\psi - \sin\phi \sin\psi & -\cos\phi \cos\theta \sin\psi - \sin\phi \cos\psi & \cos\phi \sin\theta \\ \sin\phi \cos\theta \cos\psi + \cos\phi \sin\psi & -\sin\phi \cos\theta \sin\psi + \cos\phi \cos\psi & \sin\phi \sin\theta \\ -\sin\theta \cos\psi & \sin\theta \sin\psi & \cos\theta \end{bmatrix}. \quad (2.20)$$

Let $\mathbf{R}(\hat{W}, \alpha)$ denote the rotation matrix between a coordinate frame and the frame obtained by its rotation about an axis \hat{W} by an angle α . Then equation 2.20 can also be written as

$${}^A\mathbf{R}_B = \mathbf{R}(Z_A, \phi) \mathbf{R}(Y_{A'}, \theta) \mathbf{R}(Z_{A''}, \psi). \quad (2.21)$$

Hence we have shown that the rotation matrix equivalent to any Euler angles (ϕ, θ, ψ) is given uniquely by equation 2.20.

Now consider the problem of obtaining the Euler angles for a given ${}^A\mathbf{R}_B$. Assume that

$${}^A\mathbf{R}_B = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \quad (2.22)$$

is given. Equating each element of equation 2.20 and equation 2.22, we have

$$\cos\phi \cos\theta \cos\psi - \sin\phi \sin\psi = R_{11}, \quad (2.23a)$$

$$-\cos\phi \cos\theta \sin\psi - \sin\phi \cos\psi = R_{12}, \quad (2.23b)$$

$$\cos\phi \sin\theta = R_{13}, \quad (2.23c)$$

$$\sin\phi \cos\theta \cos\psi + \cos\phi \sin\psi = R_{21}, \quad (2.23d)$$

$$-\sin\phi \cos\theta \sin\psi + \cos\phi \cos\psi = R_{22}, \quad (2.23e)$$

$$\sin\phi \sin\theta = R_{23}, \quad (2.23f)$$

$$-\sin\theta \cos\psi = R_{31}, \quad (2.23g)$$

$$\sin\theta \sin\psi = R_{32}, \quad (2.23h)$$

$$\cos\theta = R_{33}. \quad (2.23i)$$

From equations 2.23c and 2.23f,

$$\sin\theta = \pm \sqrt{R_{13}^2 + R_{23}^2}. \quad (2.24)$$

Hence, from equations 2.24 and 2.23i we obtain

$$\theta = \text{atan2}(\pm \sqrt{R_{13}^2 + R_{23}^2}, R_{33}). \quad (2.25)$$

Either the top or the bottom symbol of \pm or \mp should be used consistently throughout this section. In equation 2.25, atan2 is a scalar function defined by

$$\text{atan2}(a, b) = \arg(b + ja), \quad (2.26)$$

where j is the imaginary unit and $\arg(\cdot)$ is the argument of a complex number (see figure 2.5). The function atan2 is a kind of arc tangent function that satisfies

$$\theta = \text{atan2}(\sin\theta, \cos\theta) \quad (2.27)$$

and

$$\theta = \text{atan2}(k \sin\theta, k \cos\theta) \quad (2.28)$$

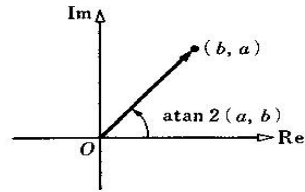


Figure 2.5
Function atan2.

for any positive real number k . Several properties of the function atan2 are also summarized in appendix 1.

Angles ϕ and ψ will be determined next. If $\sin\theta \neq 0$, we have

$$\phi = \text{atan2}(\pm R_{23}, \pm R_{13}) \quad (2.29)$$

from equations 2.23c and 2.23f, and

$$\psi = \text{atan2}(\pm R_{32}, \mp R_{31}) \quad (2.30)$$

from equations 2.23g and 2.23h. Hence, if $R_{13}^2 + R_{23}^2 \neq 0$, the Euler angles are obtained from equations 2.25, 2.29, and 2.30:

$$\phi = \text{atan2}(\pm R_{23}, \pm R_{13}), \quad (2.31a)$$

$$\theta = \text{atan2}(\pm \sqrt{R_{13}^2 + R_{23}^2}, R_{33}), \quad (2.31b)$$

$$\psi = \text{atan2}(\pm R_{32}, \mp R_{31}). \quad (2.31c)$$

It is also straightforward to show that these solutions satisfy the other equations in 2.23. Although there are two sets of Euler angles for a given ${}^A R_B$, if we put the constraint $0 < \theta < \pi$ on θ we have the following unique Euler angles:

$$\phi = \text{atan2}(R_{23}, R_{13}), \quad (2.32a)$$

$$\theta = \text{atan2}(\sqrt{R_{13}^2 + R_{23}^2}, R_{33}), \quad (2.32b)$$

$$\psi = \text{atan2}(R_{32}, -R_{31}). \quad (2.32c)$$

On the other hand, if $R_{13}^2 + R_{23}^2 = 0$, we have

$$\phi = \text{arbitrary}, \quad (2.33a)$$

$$\theta = 90^\circ - R_{33} \times 90^\circ, \quad (2.33b)$$

$$\psi = \text{atan2}(R_{21}, R_{22}) - R_{33}\phi. \quad (2.33c)$$

In other words, in the case of $\theta = 0$ or $\theta = \pi$, there is an infinite number of combinations of ϕ and ψ for just one orientation. So special care should be taken in treating this case. If we wish to keep uniqueness even in this case, one way is to set $\phi = 0$ and $\psi = \text{atan2}(R_{21}, R_{22})$.

We can also find a general expression of the Euler angles for a given rotation matrix which needs no distinction between the cases $\sin\theta = 0$ and $\sin\theta \neq 0$; see exercise 2.2.

Example 2.2 The representation by Euler angles of the orientation in example 2.1 is

$$(\phi, \theta, \psi) = (-90^\circ, 90^\circ, 90^\circ),$$

or

$$(\phi, \theta, \psi) = (90^\circ, -90^\circ, -90^\circ). \quad \square$$

2.1.4 Roll, Pitch, and Yaw Angles

The concept of roll, pitch, and yaw angles is basically the same as that of Euler angles except for a difference in the way we select the third rotational axis. In the case of roll, pitch, and yaw angles (figure 2.6),

(i) the frame obtained from Σ_A by a rotation about Z_A by ϕ is named $\Sigma_{A'}$, as with Euler angles,

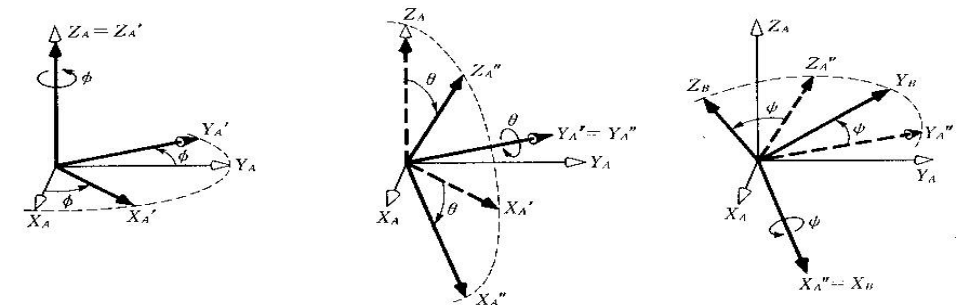


Figure 2.6
Roll, pitch, and yaw angles (ϕ, θ, ψ).

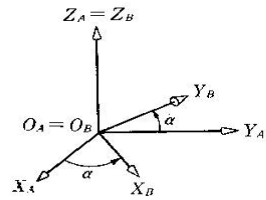


Figure 2.8
Frames Σ_A and Σ_B .

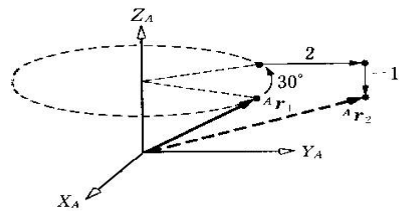


Figure 2.9
Transfer of a point.

$${}^A T_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \square$$

(iii) *Description of transfer of a point in space.* We assume that a point in space with a fixed coordinate frame is transferred in a given way. The homogeneous transform can be used to describe the relation between the original point location and the location after the transfer.

Example 2.6 As shown in figure 2.9, we consider a point ${}^A r_1$ expressed in Σ_A . Let ${}^A r_2$ denote the new location of this point after rotating 30° about Z_A and then translating 2 units along Y_A and -1 units along Z_A . The relation between ${}^A r_1$ and ${}^A r_2$ is described by

$${}^A r_2 = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 0 & 0 \\ \sin 30^\circ & \cos 30^\circ & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^A r_1 \cdot \square$$

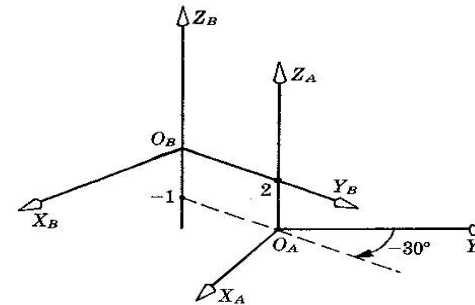


Figure 2.10
Frames Σ_A and Σ_B .

These three uses of homogeneous transformation are of course closely related, as example 2.7 will show.

Example 2.7 When the transform given in example 2.6 is regarded as ${}^B T_A$ describing the relation between two frames Σ_A and Σ_B , we have the change in frame shown in figure 2.10. Note that Σ_B is obtained from a frame originally at the same place as Σ_A , first by translating it -2 units along Y_A and 1 unit along Z_A and then by rotating it -30° about Z_A . Also note that the point whose representation in Σ_A is ${}^A r_1$ is represented as ${}^A r_2$ in Σ_B . \square

2.2.2 Product and Inverse of Homogeneous Transform

Consider three frames, Σ_A , Σ_B , and Σ_C , and assume that the relations between Σ_A and Σ_B and between Σ_B and Σ_C are given by ${}^A T_B$ and ${}^B T_C$ respectively. Then the relation ${}^A T_C$ between Σ_A and Σ_C is given by the product of ${}^A T_B$ and ${}^B T_C$:

$${}^A T_C = {}^A T_B {}^B T_C. \tag{2.37}$$

This is an extension of equation 2.13.

The following two interpretations in terms of coordinate-frame transformations are possible for equation 2.37:

- (i) The frame Σ_B is obtained from Σ_A by translating Σ_A by ${}^A p_B$ with respect to Σ_A and then rotating the translated frame (using ${}^A R_B$) about the origin of the translated frame itself. The frame Σ_C is obtained from Σ_B by translating by ${}^B p_C$ and then rotating the translated frame (using ${}^B R_C$) about the origin of the translated frame.

(ii) We denote as Σ_B the frame obtained by first rotating a frame initially coincident with Σ_A using ${}^B R_C$ and then translating it by ${}^B p_C$, both with respect to Σ_A . Further, we rotate Σ_B using ${}^A R_B$ and translate it by ${}^A p_B$, both with respect to Σ_A . Thus we obtain the frame Σ_C .

In interpretation i, the product of two transforms is interpreted from the left one (${}^A T_B$), and the second transformation (${}^B T_C$) is done with respect to the new frame obtained from the left transformation. In contrast, in interpretation ii the product is interpreted from the right transform (${}^B T_C$), and both transformations are done with respect to the original frame. The difference of order between the translation and the rotation in interpretations i and ii could be understood by substituting into equation 2.37 the relation

$${}^A T_B = \begin{bmatrix} {}^A R_B & | & {}^A p_B \\ \hline \mathbf{0} & | & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & | & {}^A p_B \\ \hline \mathbf{0} & | & 1 \end{bmatrix} \begin{bmatrix} {}^A R_B & | & \mathbf{0} \\ \hline \mathbf{0} & | & 1 \end{bmatrix}, \quad (2.38)$$

and a similar relation for ${}^B T_C$, where $\mathbf{0}$ denotes a vector or a matrix whose elements are all zeros.

Example 2.8 Assume that ${}^A T_B$ and ${}^B T_C$ are given by

$${}^A T_B = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 & 2 \\ 1/2 & \sqrt{3}/2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$${}^B T_C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The two interpretations of ${}^A T_C$ are illustrated schematically in figure 2.11. In this example the Z axis is always normal to the page and thus it is omitted from the figure. The final location of Σ_C is, of course, the same in both cases.

□

The inverse of ${}^A T_B$ is, from equation 2.4,

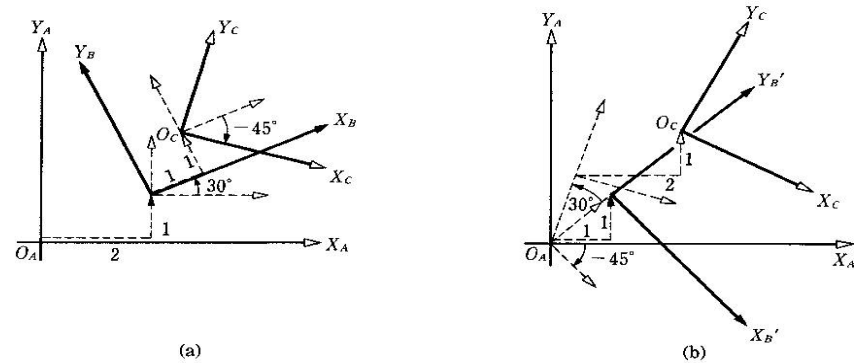


Figure 2.11 Interpretation of ${}^A T_C$. (a) Interpretation by (i). (b) Interpretation by (ii).

$${}^B T_A = {}^A T_B^{-1} = \begin{bmatrix} {}^A R_B^T & | & -{}^A R_B^T {}^A p_B \\ \hline \mathbf{0} & | & 1 \end{bmatrix}. \quad (2.39)$$

2.3 Joint Variables and Position of End Effector

2.3.1 General Relation

In this subsection an overview is given of the relation between joint displacements (rotational displacements for revolute joints and linear displacements for prismatic joints) and end-effector position for a manipulator with n degrees of freedom.

In figure 2.12 the joints are numbered $1, 2, \dots, n$, starting from the base of the manipulator. The displacement of joint i is denoted q_i and is called the *joint variable*. The collection of joint variables

$$\mathbf{q} = [q_1, q_2, \dots, q_n]^T \quad (2.40)$$

is called the *joint vector*. The position of the end effector is denoted by the m -dimensional vector

$$\mathbf{r} = [r_1, r_2, \dots, r_m]^T, \quad (2.41)$$

where $m \leq n$. For a general case where the end effector can take an arbitrary position and orientation in three-dimensional Euclidean space, we have

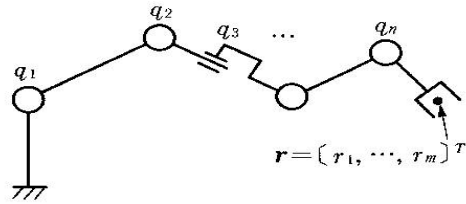


Figure 2.12
n-link manipulator.

$m = 6$. However, when the manipulator moves in a two-dimensional plane and we are concerned only with its endpoint position in the plane, we can set $m = 2$. If we are further concerned with the orientation of the endpoint in the plane, we have to set $m = 3$.

The relation between r and q , determined by the manipulator mechanism, generally is nonlinear. We assume that this relation is given by

$$r = f_r(q). \tag{2.42}$$

This equation is called the *kinematic equation* of the manipulator. When the joint vector q is given, the corresponding r is determined uniquely and the calculation is rather simple. However, when some task is assigned to the manipulator, what is given first is usually its end-effector position r or a trajectory of r . Thus we have to calculate a joint vector q which will realize the required end-effector position r —that is, we have to obtain q satisfying equation 2.42. This solution can be written formally as

$$q = f_r^{-1}(r). \tag{2.43}$$

Note, however, that q does not necessarily exist, and even when it does exist it may not be unique. The problem of obtaining r for a given q is called the *direct kinematics problem*, and that of obtaining q corresponding to a given r is called the *inverse kinematics problem*. As we can see from the above argument, the inverse kinematics problem is usually the more difficult of the two.

Example 2.9 The relations 2.42 and 2.43 will be obtained for the two-degrees-of-freedom manipulator moving in the X - Y plane as shown in figure 2.13. The joint vector in this case is $q = [\theta_1, \theta_2]^T$, the end-effector position is given by $r = [x, y]^T$, and the lengths of the links are l_1 and l_2 . The relation 2.42 is easily obtained:

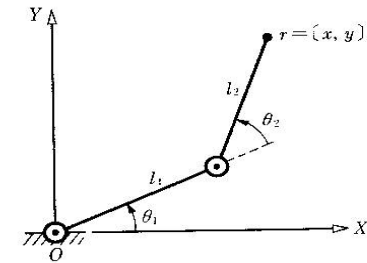


Figure 2.13
Two-link manipulator.

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2),$$

$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2).$$

As for the relation 2.43, if r satisfies

$$(l_1 - l_2)^2 \leq x^2 + y^2 \leq (l_1 + l_2)^2$$

then there exists at least one q corresponding to r . Further, if $x^2 + y^2 \neq 0$, using the function atan2 , from equations A1.11, A1.4, and A1.6 of appendix 1, we obtain

$$\theta_1 = \text{atan2}(y, x) \mp \text{atan2}(\kappa, x^2 + y^2 + l_1^2 - l_2^2)$$

and

$$\theta_2 = \pm \text{atan2}(\kappa, x^2 + y^2 - l_1^2 - l_2^2),$$

where

$$\kappa = \sqrt{(x^2 + y^2 + l_1^2 + l_2^2)^2 - 2[(x^2 + y^2)^2 + l_1^4 + l_2^4]}.$$

The two solutions given above correspond to the two arm configurations in figure 2.14. If, on the other hand, $x^2 + y^2 = (l_1 - l_2)^2 = 0$, then

$$\theta_1 = \text{arbitrary}$$

and

$$\theta_2 = \pm 180^\circ;$$

that is, there are infinitely many solutions. \square

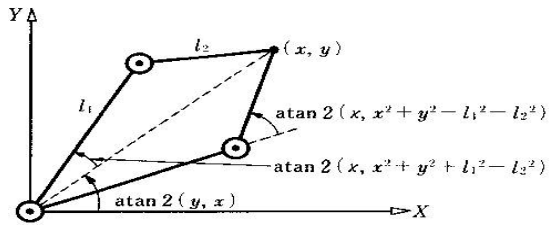


Figure 2.14
Solutions of inverse kinematics problem for two-link manipulator.

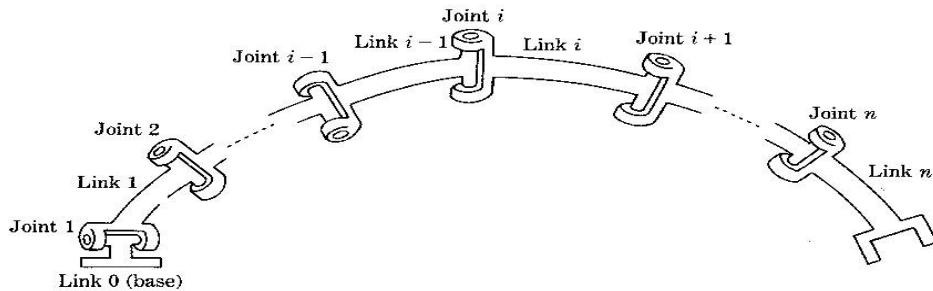


Figure 2.15
Link and joint numbers for n -link manipulator.

As example 2.9 shows, it is rather easy to solve the direct kinematics problem and the inverse kinematics problem for manipulators with around two degrees of freedom. However, both problems become increasingly difficult as the number of degrees of freedom increases. One way to cope with this difficulty is to assign an appropriate coordinate frame to each link and to describe the relation among the links by the relation among these frames. In this way, deriving the function $f_r(q)$ becomes systematic and obtaining $f_r^{-1}(r)$ analytically or numerically becomes much easier. This approach is developed in the following subsections.

2.3.2 Link Parameters

Consider a manipulator consisting of n links connected serially by n joints, with one degree of freedom each, which may be either revolute or prismatic. As is shown in figure 2.15, the links and the joints are numbered 1, ..., n ,

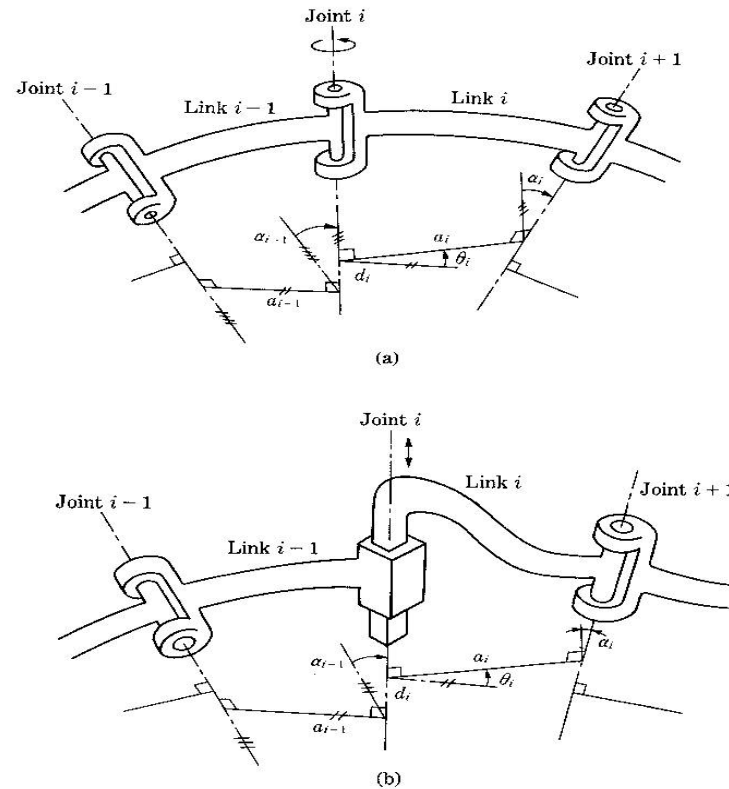


Figure 2.16
Joint axes, joint variables, and link parameters: (a) when joint i is revolute; (b) when joint i is prismatic.